A Newton's Variant Third Order Method

V.B. Kumar Vatti[1] , Ramadevi Sri[1]
[1]Department of Engineering Mathematics,
Andhra University, Visakhapatnam, Andhra Pradesh, India
Email: drvattivbk@yahoo.co.in

M.S.Kumar Mylapalli[2]
[2]Department of Mathematics,
GITAM University, Visakhapatnam, Andhra Pradesh, India
Email: maniraj9@yahoo.co.in

Abstract: **In this paper, we present a new iterative method to solve the nonlinear equation $f(x)=0$ and discuss about its convergence. Few numerical examples are considered to show the efficiency of the new method in comparison with the other methods considered in this paper.**

***Keywords*: Nonlinear equation, Iterative method, Newton's method, Convergence.**

I. INTRODUCTION

One of the most important problems in numerical analysis is solving nonlinear equation

$$f(x)=0 \qquad (1.1)$$

Where $f: I \rightarrow R$ for an open interval I is a scalar function.

Let x_{n+1} be the root of the equation (1.1) i.e., $f(x_{n+1})=0$ while $f'(x_{n+1}) \neq 0$.

There are many methods developed on the improvement of quadratically convergent Newton' s Method so as to get a superior convergence rate than this. Earlier, many investigations [1,6,8,9] have made to explain the root of nonlinear algebraic and transcendental equation. For the same purpose the variants of the Newton' s formulas have been discussed by Babajee and Dauhoo [5], whereas other [3,4] suggests the multi-step iterative method for it.

The classical quadratic convergent Newton's method [12] for finding the root of (1.1) is

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)} \qquad (1.2)$$

$$(n=0,1,2,\ldots)$$

The fourth order method proposed by Maheshweri [11], is

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1}=x_n+\frac{1}{f'(x_n)}\left[\frac{f^2(x_n)}{f(y_n)-f(x_n)}-\frac{f^2(y_n)}{f(x_n)}\right] \qquad (1.3)$$

$$(n=0,1,2,\ldots)$$

The new hybrid iteration method proposed by Nasr-Al-Din Ide [2], is

$$x_{n+1}=\frac{-B\pm\sqrt{B^2-4\cdot A\cdot C}}{2A} \qquad (1.4)$$

$$(n=0,1,2,\ldots)$$

Where, $A=f''(x_n)$, $B=6f'(x_n)-2f''(x_n)x_n$ and $C=6f(x_n)-6f'(x_n)x_n+f''(x_n)x^2{}_n$.

In this paper, we present a Newton's Variant iterative method is given in section 2. In section 3, the convergence criterion of the new method is discussed where as in the concluding section several numerical examples are considered to exhibit the efficiency of the developed method.

II. NEWTON'S VARIANT THIRD ORDER METHOD

Following the basic assumption of Abbasbandy and Maheshweri [1, 11] and also others [7, 13], we consider the second degree Taylor's expansion of $f(x_{n+1})$ about x_n is

$$f(x_{n+1})=f(x_n)+(x_{n+1}-x_n)f'(x_n)+\frac{(x_{n+1}-x_n)^2}{2}f''(x_n) \qquad (2.1)$$

Where $x_{n+1}-x_n=h$

$$f(x_{n+1})=x^2{}_{n+1}\left[\frac{f''(x_n)}{2}\right]+x_{n+1}\left[f'(x_n)-x_nf''(x_n)\right]+\left[f(x_n)-x_nf'(x_n)+\frac{x_n{}^2f''(x_n)}{2}\right] \qquad (2.2)$$

Since, x_{n+1} be the root of the equation (1.1) i.e., $f(x_{n+1})=0$ then the equation (2.2) becomes

$$x^2{}_{n+1}\left[f''(x_n)\right]+x_{n+1}\left[2f'(x_n)-2x_nf''(x_n)\right]+\left[2f(x_n)-2x_nf'(x_n)+x_n{}^2f''(x_n)\right]=0 \qquad (2.3)$$

Solving for x_{n+1} from (2.3), we get

$$x_{n+1}=\frac{\left[x_nf''(x_n)-f'(x_n)\right]\pm\sqrt{(f'(x_n))^2-2f(x_n)f''(x_n)}}{f''(x_n)} \qquad (2.4)$$

The iterative scheme from the equation (2.4) is

$$x_{n+1}=x_n-\frac{f'(x_n)}{f''(x_n)}\left[1-\sqrt{1-2\omega_n}\right] \qquad (2.5)$$

Where $\omega_n=\frac{f(x_n)f''(x_n)}{\left[f'(x_n)\right]^2}$ (2.6)

The required new iteration scheme is obtained by rewriting equation (2.5) as

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}\left[\frac{2}{1+\sqrt{1-2\omega_n}}\right] \qquad (2.7)$$

$$(n=0,1,2,\ldots)$$

where, ω_n is as given in (2.6).

III. CONVERGENCE CRITERIA

Theorem 3.1. Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f: I \to R$ for an open interval *I*. Then, the new method that is defined by equation (2.6) has the third order convergence and satisfies the following error equation,

$$\varepsilon_{n+1} = c_3 \varepsilon_n^3 + o\left(\varepsilon_n^4\right)$$

Where, $x_{n+1} = \varepsilon_{n+1} + \alpha$

Proof: Let α be a simple zero of equation (1.1). By the Taylor's expansions

$$f\left(x_n\right) = f'(\alpha)\left[\varepsilon_n + c_2\varepsilon_n^2 + c_3\varepsilon_n^3 + o\left(\varepsilon_n^4\right)\right] \quad (3.1)$$

and $$f'\left(x_n\right) = f'(\alpha)\left[1 + 2c_2\varepsilon_n + 3c_3\varepsilon_n^2 + 4c_4\varepsilon_n^3 + o\left(\varepsilon_n^4\right)\right] \quad (3.2)$$

$$\left[f'\left(x_n\right)\right]^2 = \left[f'(\alpha)\right]^2 \bullet \left[1 + 4c_2\varepsilon_n + \left(6c_3 + 4c_2^2\right)\varepsilon_n^2 + \left(12c_2c_3 + 8c_4\right)\varepsilon_n^3 + o\left(\varepsilon_n^4\right)\right] \quad (3.3)$$

$$f''\left(x_n\right) = f'(\alpha) \bullet \left[2c_2 + 6c_3\varepsilon_n + 12c_4\varepsilon_n^2 + 20c_5\varepsilon_n^3 + o\left(\varepsilon_n^4\right)\right] \quad (3.4)$$

Multiplying the equations (3.1) and (3.4), we get

$$f\left(x_n\right) f''\left(x_n\right) = \left[f'(\alpha)\right]^2 \bullet \left[2c_2\varepsilon_n + \left(6c_3 + 2c_2^2\right)\varepsilon_n^2 + \left(8c_2c_3 + 12c_4\right)\varepsilon_n^3 + o\left(\varepsilon_n^4\right)\right] \quad (3.5)$$

Dividing the equation (3.5) by (3.3), we obtain

$$\omega_n = \frac{f\left(x_n\right) f''\left(x_n\right)}{\left[f'\left(x_n\right)\right]^2} = 2c_2\varepsilon_n + \left(6c_3 - 6c_2^2\right)\varepsilon_n^2 + \left(16c_2^3 - 28c_2c_3 + 12c_4\right)\varepsilon_n^3 + o\left(\varepsilon_n^4\right) \quad (3.6)$$

Again dividing the equation (3.1) by (3.2), we have

$$\frac{f\left(x_n\right)}{f'\left(x_n\right)} = \varepsilon_n - c_2\varepsilon_n^2 + 2\left(c_2^2 - c_3\right)\varepsilon_n^3 + o\left(\varepsilon_n^4\right) \quad (3.7)$$

Now by the equation (3.6), we obtain

$$\left(1 - 2\omega_n\right)^{\frac{1}{2}} = 1 - 2c_2\varepsilon_n + \left(4c_2^2 - 6c_3\right)\varepsilon_n^2 + \left(16c_2c_3 - 8c_2^3 - 12c_4\right)\varepsilon_n^3 + o\left(\varepsilon_n^4\right) \quad (3.8)$$

and

$$\left[1 + \left(1 - 2\omega_n\right)^{\frac{1}{2}}\right]^{-1} = 2^{-1}\left[1 + c_2\varepsilon_n + \left(3c_3 - c_2^2\right)\varepsilon_n^2 + \left(c_2^3 + 6c_4 - 2c_2c_3\right)\varepsilon_n^3 + o\left(\varepsilon_n^4\right)\right] \quad (3.9)$$

From the equations (3.7) and (3.9), we get

$$\frac{f\left(x_n\right)}{f'\left(x_n\right)}\left[\frac{2}{1 + \sqrt{1 - 2\omega_n}}\right] = \varepsilon_n + c_3\varepsilon_n^3 + o\left(\varepsilon_n^4\right) \quad (3.10)$$

Thus, $x_{n+1} = x_n - \frac{f\left(x_n\right)}{f'\left(x_n\right)}\left[\frac{2}{1 + \sqrt{1 - 2\omega_n}}\right]$ becomes

$$\varepsilon_{n+1} + \alpha = \left(\varepsilon_n + \alpha\right) - \varepsilon_n + c_3\varepsilon_n^3 + o\left(\varepsilon_n^4\right)$$

$$\varepsilon_{n+1} = c_3\varepsilon_n^3 + o\left(\varepsilon_n^4\right) \quad (3.11)$$

Equation (3.11) establishes the third order convergence of the method that is defined by equation (2.7).

IV. NUMERICAL EXAMPLES

We consider few numerical examples considered by [2, 13] and the method (2.7) is compared with the methods (1.2), (1.3) and (1.4). The computational results are tabulated below and the results are correct up to an error less than ε as indicated for each of the problems.

Example 1. Consider the following equation [2, 13],
$f(x) = x^3 - e^{-x} = 0$

TABLE 1.
The results obtained by four methods for solving $f(x) = x^3 - e^{-x} = 0$ with $x_0 = 1$ and $\varepsilon = 0.5E-16$

Formula	n	x_n	No. of functional evaluations
Newton	6	0.7728829591492102	12
Hybrid	6	0.7728829591492101	18
New Hybrid	6	0.7728829591492101	18
New Iteration	4	0.7728829591492102	12

Example 2. Consider the following equation [2, 13],
$f(x) = \sin x - 0.5x = 0$

TABLE 2.
The results obtained by four methods for solving $f(x) = \sin x - 0.5x = 0$ with $x_0 = 1.6$ and $\varepsilon = 0.5E-15$

Formula	n	x_n	No. of functional evaluations
Newton	4	1.895494267033981	8
Hybrid	6	1.895494267033999	18
New Hybrid	5	1.895494267033999	15
New Iteration	4	1.895494267033981	12

Example 3. Consider the following equation [2, 13],
$f(x) = x \ln x - 1.2 = 0$

TABLE 3.
The results obtained by four methods for solving $f(x) = x \ln x - 1.2 = 0$ with $x_0 = 2$ and $\varepsilon = 0.5E-11$

Formula	n	x_n	No. of functional evaluations
Newton	4	1.88808675303	8
Hybrid	10	1.88808675303	30
New Hybrid	6	1.88808675303	18
New Iteration	3	1.88808675303	9

Example 4. Consider the following equation [2, 13],
$f(x) = \tan^{-1} x = 0$

TABLE 4.
The results obtained by four methods for solving $f(x) = \tan^{-1} x = 0$ with $x_0 = 2$ and $\varepsilon = 0.5E-18$

Formula	n	x_n	No. of functional evaluations
Newton	11	Failure	---
Hybrid	6	0	18
New Hybrid	7	0	21
New Iteration	5	0	15

Example 5. Consider the following equation [2, 13],

$f(x) = x^3 + 4x^2 - 10 = 0$

The results obtained by four methods for solving $f(x) = x^3 + 4x^2 - 10 = 0$ with $x_0 = 1.5$ and $\varepsilon = 0.5E - 8$

TABLE 5.

Formula	n	x_n	No. of functional evaluations
Newton	4	1.36523001	8
Hybrid	6	1.36523001	18
New Hybrid	4	1.36523001	12
New Iteration	3	1.36523001	9

V. CONCLUSION

With the number of iterations and the number of functional evaluations tabulated for each of the methods for five non-linear equations, we conclude that the method (2.7) is efficient one compared to the methods considered in this paper.

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AUTHOR'S PROFILE

V.B. Kumar, Vatti received the Doctoral of Philosophy in Mathematics from Indian Institute of Technology, Mumbai in 1987. Presently he is working as a Professor of Engineering Mathematics in Andhra University. He has nearly above 30 years of teaching experience and his research interest includes developing application oriented numerical methods.

Ramadevi Sri received the Master's degree in Mathematics from Andhra University, Visakhapatnam in 2009. Currently, she is pursuing for Ph.D. degree in Engineering Mathematics in Andhra University. Her research interest includes developing numerical methods for solving non-linear equations.

M.S. Kumar, Mylapalli received the Doctoral of Philosophy in Applied Mathematics from Andhra University, Visakhapatnam in 2010. Presently he is working as a Assistant Professor of Mathematics in Gitam University. He has nearly above 11 years of teaching experience and his research interest includes developing numerical methods for solving nonlinear equation, system of linear and nonlinear equations.