On Characteristics of Wrapped Gamma Distribution

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***Abstract* - The Gamma distribution is fitted to the amounts of daily rainfall and to the differences between the flows of successive days on the ascension curve of the hydrograph [c.f. Hafzullah Aksoy (2000)]. In the study of weather conditions and wind directions Circular distributions will yield suitable results. Dattatreya Rao et al (2007) constructed a good number of wrapped circular models by applying the method of wrapping. Viewing in this aspect here we made an attempt to generate a new circular model coined as Wrapped Gamma Distribution by employing the wrapping approach, the most popular method of constructing circular models. The trigonometric moments derived from the characteristic function are used to evaluate the characteristics of population of the new circular model. The graphs of the probability density function are plotted with the help of MATLAB code.**

***Keywords*: Circular model, wrapping, trigonometric moments, Wrapped Gamma distribution, characteristics.**

1. INTRODUCTION

In many diverse fields, the measurements are directions- A biologist may be interested in the direction of flight of the bird or orientation of an animal while geologist may be measuring the direction of earth's magnetic field. Such directions may be in two or three dimensions. A set of such observations on the directions is referred to as 'directional data'. In particular analysis pertaining to two dimensional directional data falls under the topic **'CIRCULAR STATISTICS'**. For such data, several Statistical models were constructed and inference procedures were studied.

Dattatreya Rao et al (2007) have introduced Wrapped versions of four well known life testing models viz .Lognormal, Logistic, Weibull and Extreme-Value distributions and mentioned basic characteristics along with graphs of the above said distributions. Here an attempt is made on the lines of Dattatreya Rao et al (2007) to derive a new circular model coined as Wrapped Gamma distribution by reducing a linear variable to it's modulo 2π and using trigonometric moments. Certain population characteristics are also studied.

In the continuous case $g:[0,2\pi)\rightarrow \mathbb{R}$ is the probability density function of a circular distribution iff g has the following basic properties

- $g(\theta) \geq 0, \quad \forall\theta$ (1.1)
- $\int_0^{2\pi} g(\theta)\, d\theta = 1$ (1.2)
- $g(\theta) = g(\theta + 2k\pi)$ (1.3)

for any integer k (i.e., g is periodic)

(Mardia,1972 and Jammalamadaka, Sengupta 2001)

2. METHODOLOGY OF WRAPPING

Modulo 2π reduction

If X is a r.v. defined on $\mathbb{R}$, then the corresponding circular r.v. X_W is defined by the modulo 2π reduction.

$$X_W \equiv X\left(\text{mod}\, 2\pi\right) \quad (2.1)$$

It is clearly a many valued function given by

$$X_W(\theta) = \{X(\theta + 2k\pi) / k \in \mathbb{Z}\} \quad (2.2)$$

The wrapped circular pdf $g(\theta)$ corresponding to the density function f of a linear r.v. X is defined as,

$$g(\theta) = \sum_{k=-\infty}^{\infty} f(\theta + 2k\pi),\ \theta \in [0, 2\pi) \quad (2.3)$$

It may be noted that the circular distribution is a probability distribution whose total probability is concentrated on the unit circle $\{(\cos\theta, \sin\theta) / 0 \le \theta < 2\pi\}$ in the plane which satisfies the properties (2.1) through (2.3).

3. WRAPPED GAMMA DISTRIBUTION

The gamma distribution is prescribed by two parameters, one for scale and one for shape. A continuous random variable X with density function [Johnson and Kotz (2000)] where a is scale and λ is shape parameters.

$$f(x) = \begin{cases} \dfrac{a^{\lambda} e^{-ax} x^{\lambda-1}}{\Gamma(\lambda)}, 0 < x < \infty \\ 0,\ \text{otherwise and} \\ a > 0,\ \lambda > 0 \end{cases}$$

The cdf and the Characteristic function of the Gamma distribution are

$$F(x) = \frac{\Gamma_x(\lambda)}{\Gamma(\lambda)}$$

$$\phi_x(t) = \frac{a^{\lambda}}{(a - it)^{\lambda}},\ \text{where } t \in \mathbb{R}$$

By applying the method of wrapping we get the corresponding circular model Wrapped Gamma model.

The pdf and cdf of the Wrapped Gamma model are

$$g(\theta) = \sum_{k=0}^{\infty} f(\theta + 2k\pi) = \sum_{k=0}^{\infty} \frac{a^{\lambda} e^{-a(\theta+2k\pi)} (\theta + 2k\pi)^{\lambda-1}}{\Gamma(\lambda)}, \theta \in [0, 2\pi)$$

where $a > 0, \lambda > 0$

$$G(\theta) = \sum_{k=0}^{\infty} \left\{F(\theta + 2k\pi) - F(2k\pi)\right\} = \sum_{k=0}^{\infty} \left\{\frac{\Gamma_{\theta+2k\pi}(\lambda)}{\Gamma(\lambda)} - \frac{\Gamma_{2k\pi}(\lambda)}{\Gamma(\lambda)}\right\}$$

If $G(\theta)$ denotes the cdf of the r.v., the characteristic function of the circular model is given by

$$\varphi_{\theta}(t) = E\left(e^{it\theta}\right) = \int_0^{2\pi} e^{it\theta} dG(\theta) = \rho_t e^{i\mu_t} \quad t \in \mathbb{R}$$

It can be seen that whenever $\varphi(t) \ne 0$, $e^{2\pi it} = 1$ (p. 41 Mardia 1972). This suggests that the function $\varphi(t)$ should only be defined for integer values of *t*. Accordingly the characteristic function $\varphi(p) = \varphi_p$ is defined by

$$\varphi_{\theta}(p) = E\left(e^{ip\theta}\right) = \int_0^{2\pi} e^{ip\theta} dF(\theta) = \rho_p e^{i\mu_p}$$

$$p \in \mathbb{Z}$$

$$\text{Clearly, } \phi_0 = 1, \quad \bar{\phi}_p = \phi_{-p}.$$

The Characteristic function of the Wrapped Gamma model is

$$\phi_{\Phi}(p) = \frac{a^{\lambda}}{(a - ip)^{\lambda}},\ \text{where } p \in \mathbb{Z}$$

Trigonometric moments

The value of the characteristic function φ_p at an integer p is also called the p^{th} trigonometric moment of θ. The real part and the imaginary part of φ_p are denoted by α_p and β_p respectively. We can also view these trigonometric moments in terms of

$$\alpha_p = E(\cos p\theta), \quad \beta_p = E(\sin p\theta), \quad p \in \mathbb{Z}$$

The first trigonometric moment namely,

$\varphi_1 = \alpha_1 + i\beta_1 = \rho_1 e^{i\mu_1}$ plays a prominent role in determining the mean direction and resultant length.

The pdf of a wrapped circular model can be obtained through characteristic function of the linear r.v. X using trigonometric moments. Using the inversion theorem of characteristic function, one can derive, circular models through trigonometric moments. These trigonometric moments can be obtained using the following Proposition [c.f. p.31, Rao Jammalamadaka, and Sen Gupta (2001)] and Carslaw(1930).

Proposition *The trigonometric moment of order* p *for a wrapped circular distribution corresponds to the value of the characteristic function of the unwrapped r.v.* X *, say* $\varphi_p = \phi_X(p)$ *for* $p \in \mathbb{Z}$.

If α_p and β_p are the trigonometric moments and $\sum_{p=1}^{\infty}\left(\alpha_p^2 + \beta_p^2\right)$ is convergent then the random variable θ has a density g which is defined **almost everywhere** by

$$g(\theta) = \frac{1}{2\pi}\sum_{-\infty}^{\infty}\phi_p e^{-ip\theta}$$

$$= \frac{1}{2\pi}\left[1 + 2\sum_{p=1}^{\infty}(\alpha_p \cos p\theta + \beta_p \sin p\theta)\right],$$

$$\theta \in [0, 2\pi), \; p \in \mathbb{Z}$$

Definition: A r. v. Φ on the unit circle is said to have Wrapped Gamma distribution with parameter $\lambda > 0$, denoted by WG(λ), if the characteristic function and the pdf of Φ are given as above respectively. We then write $\Phi \sim$ WG(λ).

4. GRAPHS

The graphs for the pdf and the cdf of the Wrapped Gamma distribution are drawn using MATLAB and are presented here.

Figure. 1 Graph of pdf of Wrapped Gamma model

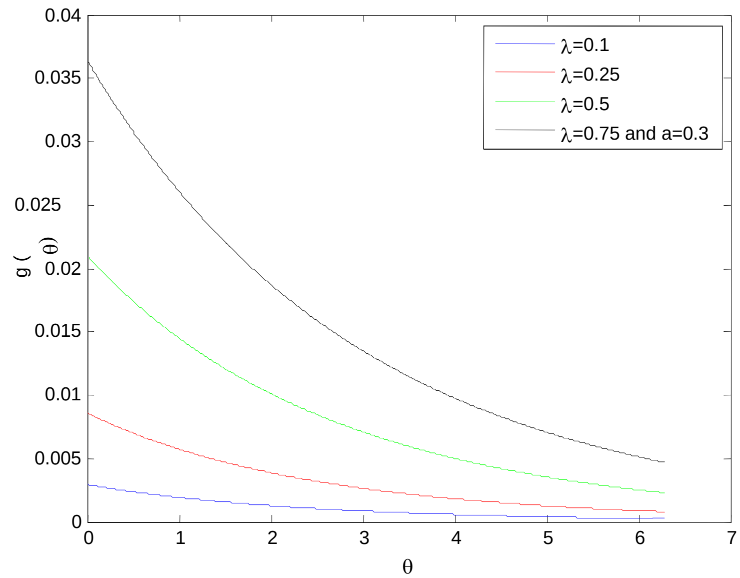
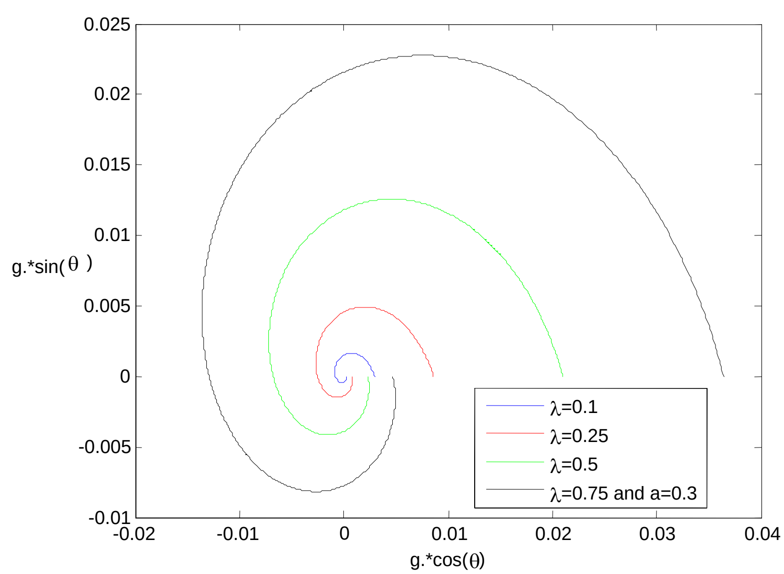


Figure. 2 Graph of pdf of Wrapped Gamma model of circular representation



5. CHARACTERISTICS OF WRAPPED GAMMA DISTRIBUTION

Parameters are to be estimated to study the characteristics of a population of a linear model whereas in case of circular models trigonometric moments which are real and imaginary parts of the characteristic function will suffice. Mardia (2000) gave expressions of mean direction, resultant length, circular variance, circular standard deviation, central trigonometric moments, skewness and kurtosis for circular distributions. These characteristics for the Wrapped Gamma model are also based on their respective trigonometric moments. These can be expressed in terms of trigonometric moments α_p and β_p and are presented here.

Table 1: Characteristics of Wrapped Gamma distribution

Wrapped Gamma distribution	$\lambda = 2$, $a = 2$	$\lambda = 3$, $a = 2$	$\lambda = 4$, $a = 2$	$\lambda = 5$, $a = 2$
Mean μ	0.9273	1.3909	4.9962	5.4598
Trigonometric Moments α_1	0.4800	0.1280	-0.1792	-0.3891
α_2	0	-0.2500	-0.2500	-0.1250
β_1	0.6400	0.7040	0.6144	0.4198
β_2	0.5000	0.2500	0	-0.1250
Resultant length ρ	0.8000 0.5000	0.7155 0.3536	0.6400 0.2500	0.5724 0.1768
Variance V_o	0.2000	0.2845	0.3600	0.4276
Central Trigonometric Moments α_1^*	0.8000	0.7155	0.6400	0.5724
α_2^*	0.4800	-0.3220	-0.2108	-0.1341
β_1^*	0	0	0	0
β_2^*	-0.1400	0.1460	0.1344	0.1152
Skewness γ_1^o	-1.5652	0.9623	0.6222	0.4119
Kurtosis γ_2^o	1.7600	-7.2191	-2.9211	-1.3210
Circular standard deviation σ_o	0.6680 1.1774	0.8182 1.4420	0.9448 1.6651	1.0563 1.8616

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